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Free Differential Algebras and Pure Spinor Action in IIB Superstring Sigma Models

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Abstract

In this paper we extend to the case of IIB superstring sigma models the method proposed in hep-th/10023500 to derive the pure spinor approach for type IIA sigma models. In particular, starting from the (Free) Differential Algebra and superspace parametrization of type IIB supergravity, extended to include the BRST differential and all the ghosts, we derive the BRST transformations of fields and ghosts as well as the standard pure spinor constraints for the ghosts λ related to supersymmetry. Moreover, using the method first proposed by us, we derive the pure spinor action for type IIB superstrings in curved supergravity backgrounds (on shell), in full agreement with the action first obtained by Berkovits and Howe.

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1   Introduction

Since bimillennium, Berkovits, sometimes together with collaborators, developed a new formalism for superstrings [1] - [8], based on the concept of pure spinor [9], [10]. It is a superspace approach, like the Green-Schwarz (G-S) one, which however replaces the κ-symmetry of the G-S formulation with a BRST symmetry where the ghosts are pure spinors. With respect to the G-S approach, it has the advantage to allow for a consistent, super Poincarè invariant quantization of the superstrings in D=10 flat background or in special backgrounds as, for instance, $AdS_5 \otimes S^5$ [7]. Moreover this formalism has the advantage over the RNS one to be able to treat fermions and R-R background fields in a natural way.

It is also interesting to extend the pure spinor approach to describe superstrings in curved supergravity backgrounds and write σ-models actions that are relevant, especially to deal with backgrounds in presence of R-R fluxes. Here the seminal paper is [11]. In this paper, the authors start from the more general classical action invariant under worldsheet conformal transformations and derive the supergravity constraints by requiring nilpotence of the BRST charge and holomorphicity of the BRST currents. An equivalent approach is to require invariance of the action under BRST charge [12].

One could also reverse this procedure: i.e., start from the geometric formulation of the relevant, ten dimensional supergravity and then derive the pure spinor action, as a modification of the G-S one, by requiring that the pure spinor action is BRST invariant. A first attempt to do that, restricted to the heterotic case, was done in [13]. Of course in order to be successful one should be able to get from the geometric formulation of the relevant supergravity model, the BRST transformations of fields and ghosts. This can be done by generalizing a procedure well known for Yang-Mills theories [14], [15] and widely applied in gauge [16], [17], [18], topological [19], [20] and supergravity theories [21]. We shall refer to this procedure as the method of Extended (Free) Differential Algebra. In addition, this method allows us to derive the pure spinor constraints for the ghosts under suitable conditions that are strongly related to the superembedding approach [24]. See also [25] where the relation between the pure spinor approach and the superembedding one was pointed out.

In [22] a version of the method of Extended Differential Algebra was applied to the case of IIA superstring σ-models in order to derive the ghost constraints and the BRST invariant action but the constraints obtained in [22] do not seem to fit with the standard pure spinor ones. In [23], one of the present authors (M.T.) has presented a variant of the method proposed in [22] that allows to derive the standard pure spinor constraints and the pure spinor action for type IIA, D=10 superstring σ-models in full agreement with [11].

In this paper we apply the approach of [23] to get the pure spinor constraints and the pure spinor action for the case of type IIB superstring σ-models in 10 dimensions.

The paper is organized as follows. In Section 2, we review the geometrical superspace formulation of type IIB supergravity [26], [27], [29], [28] and present the parametrization of torsions and curvatures by following [27]. In section 3, we write the G-S action, which is not a trivial step as it could seem, since IIB supergravity (at the classical level) is invariant under an $SL(2R)$ group, under which the NS-NS and R-R two superforms transform as a doublet.

Therefore to write the G-S action we will adopt a method proposed in [30] in a different context, which preserves formally this $SL(2R)$ symmetry. Then the G-S action suggests itself a rescaling (and a field redefinition) of the fields and superforms that describe type IIB supergravity. In section 4, we explain the method of Extended Differential Algebra and derive the pure spinor constraints for the ghosts. Moreover we present the extended parametrization of rescaled torsions and curvatures in terms of the real and imaginary parts of the spinor-like vielbeins and the other complex fields, which reduces the superspace parametrization of torsions and curvatures to a form very similar to that of IIA supergravity. Finally in section 5, we define the BRST charge and we give the BRST transformations of the antighosts and the covariant momenta $d_{\underline{\alpha}}$ (subsection 5.1) and we apply the method of [23], (and [13]), to derive the pure spinor action of type IIB superstring σ-models in full agreement with [11] (section 5.2).

2 IIB Supergravity in 10 Dimensions

2.1 Field Content and Notations

The D=10, IIB supergravity contains the following fields and forms: the vector-like superviel-beins $E^a = dZ^M E_M{}^a(Z)$, the spinor-like supervielbeins $(E^\alpha, E^{*\alpha})$, the two-superforms (B_2, B_2^*), the four-superform C_4, the chiral spinors $(\Lambda_\alpha, \Lambda_\alpha^*)$, the Lorentz superconnection Ω^{ab}, and the scalars $(V_+{}^i, V_-{}^i)$.

$V_\pm{}^i$ belong to the coset $SL(2R)/U(1)$ transforming as $V'_\pm{}^i = e^{\pm 2i\epsilon} V_\pm{}^j \Lambda_j{}^i$ with $\Lambda_j{}^i \in SL(2R)$ and $V_\pm^i$ satisfy the reality condition

$$V_+^i = \bar{V}_-^i, \qquad V_-^i = \bar{V}_+^i, \tag{2.1}$$

where, if ψ^i is an $SL(2R)$ doublet, we define

$$\bar{\psi}^i = (\tau_1 \psi^*)^i.$$

Moreover

$$\epsilon_{ij} V_+{}^i V_-{}^j = 1, \qquad (V_-{}^i V_+{}^j) - (V_-{}^j V_+{}^i) = -2\epsilon^{ij}.$$

Then one defines the one-superforms

$$2iQ = \epsilon_{ij} V_-{}^j dV_+{}^i,$$

which plays the role of $U(1)$ connection and

$$R_1 = \epsilon_{ij} V_+{}^i dV_+{}^j, \qquad R^*{}_1 = \epsilon_{ij} V_-{}^j dV_-{}^i.$$

With respect to the structure group $U(1)$, $V_\pm{}^i$ have charges $q = \pm 2$, $(E^\alpha, E^{*\alpha})$ have charges $q = \pm 1$, the two-superforms (B_2, B_2^*) have charges $q = \pm 2$, $(\Lambda_\alpha, \Lambda_\alpha^*)$ have charges $q = \pm 3$ and (R_1, R_1^*) have charges $q = \pm 4$. E^a, Ω^{ab} and C_4 are uncharged. Covariant derivatives involve the Lorentz connection $\Omega^{ab} = -\Omega^{ba}$ acting on Lorentz tensors and spinors and the connection

Q acting on charged fields. It is also convenient to introduce, at no cost, a Weyl connection Ω with zero curvature such that spinors with p upper and q lower spinorial indices have Weyl charge $p - q$. Since the Weyl curvature vanishes, Ω is a pure gauge. For instance,

$$T^a = \Delta E^a = dE^a + E^b \Omega_b{}^a,$$

$$T^\alpha = \Delta E^\alpha = dE^\alpha + E^\beta \frac{1}{4}(\Gamma^{ab})_\beta{}^\alpha \Omega_{ab} + E^\alpha \Omega - iE^\alpha Q,$$

$$\Delta \Lambda_\alpha = d\Lambda_\alpha + \frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta \Omega_{ab} \Lambda_\beta + \Omega \Lambda_\alpha - 3i\Lambda_\alpha Q,$$

etc. The Lorentz curvature is as usual

$$R^{ab} = d\Omega^{ab} + \Omega^a{}_c \Omega^{cb}.$$

Instead of the charged two-superforms (B_2, B_2^*) it is convenient to use the uncharged ones B_2^i which transform as a doublet of $SL(2R)$ and are defined as

$$B_2^i = V_-^i B_2 + V_+^i B_2^*. \tag{2.2}$$

The curvature of B_2^i is

$$H_3^i = dB_2^i,$$

and the curvature of C_4 is

$$F_5 = idC_4 + 2i\epsilon_{ij} B_2^i dB_2^j.$$

Notice that in our notations F_5 is purely imaginary.

These torsions and curvatures satisfy the Bianchi identities

$$\Delta T^a = E^b R_b{}^a, \tag{2.3}$$

$$\Delta T^\alpha = E^\beta R_\beta{}^\alpha + \frac{1}{2} R_1 R_1^* E^\alpha, \tag{2.4}$$

$$\Delta T^{*\alpha} = E^{*\beta} R_\beta{}^\alpha - \frac{1}{2} R_1 R_1^* E^{*\alpha}, \tag{2.5}$$

where $R_\alpha{}^\beta = \frac{1}{4} R^{ab}(\Gamma_{ab})_\alpha{}^\beta$. Moreover

$$\Delta R^{ab} = 0, \tag{2.6}$$

$$dH_3^i = 0, \tag{2.7}$$

$$dF_5 = -2i\epsilon_{ij} H_3^i H_3^j, \tag{2.8}$$

$$dQ = \frac{i}{2} R_1 R_1^*. \tag{2.9}$$

2.2 Parametrization of Torsions and Curvatures

We shall adopt the parametrization of Howe and West (H-W) [27], modulo some different conventions. For H-W, complex conjugation reverses the order of the factors, while for us it is simply to take the complex conjugate without reversing the order. Moreover in H-W if X is odd $E^\alpha X = -XE^\alpha$ and $(E^\alpha E^\beta)^* = -E^{*\alpha}E^{*\beta}$. In our conventions, derivatives act from right to left. Moreover objects with even grading always commute and objects with odd grading anticommute among themselves and commute with those of even grading.

If we denote with X_0 the objects in the notations of H-W, the dictionary among our notation and that of H-W is the following:

$$E^\alpha = e^{\frac{-i\pi}{4}} E_0^\alpha, \qquad E^{*\alpha} = e^{\frac{-i\pi}{4}} E_0^{*\alpha},$$

$$\Lambda^\alpha = e^{\frac{i\pi}{4}} \Lambda_0^\alpha, \qquad \Lambda^{*\alpha} = e^{\frac{i\pi}{4}} \Lambda_0^{*\alpha},$$

$$B_2 = B_{02}, \qquad C_4 = -iC_{04}.$$

Then one gets the following parametrization

$$T^a = (E^*\Gamma^a E), \tag{2.10}$$

$$T^\alpha = [\frac{1}{2}(E^*\Gamma^a E^*)(\Gamma_a\Lambda)^\alpha - E^{*\alpha}(E^*\Lambda)] + \frac{1}{16}E^a[3(E^*\Gamma^{bc})^\alpha H_{abc} + \frac{1}{3}(E^*\Gamma_{abcd})^\alpha H^{bcd}]$$

$$+E^c E^\beta[9\chi_c\delta_\beta{}^\alpha + \frac{3}{2}(\Gamma^c\Gamma^b)_\beta{}^\alpha\chi^b + \frac{1}{2}(\Gamma^{ab})_\beta{}^\alpha\chi_{abc} + \frac{1}{4}(\Gamma_c\Gamma_{abd})_\beta{}^\alpha\chi^{abd}$$

$$+\frac{1}{48}(\Gamma^{abde})_\beta{}^\alpha(\frac{1}{4}F^{(+)}_{abdec} + \chi^{(-)}_{abdec})] + \frac{1}{2}E^a E^b T_{ba}{}^\alpha, \tag{2.11}$$

$$T^{*\alpha} = [\frac{1}{2}(E\Gamma^a E)(\Gamma_a\Lambda^*)^\alpha - E^\alpha(E\Lambda^*)] + \frac{1}{16}E^a[3(E\Gamma^{bc})^\alpha H^*_{abc} + \frac{1}{3}(E\Gamma^{abcd})^\alpha H^*_{bcd}]$$

$$+E^c E^{*\beta}[-9\chi_c\delta_\beta{}^\alpha - \frac{3}{2}(\Gamma_c\Gamma_b)_\beta{}^\alpha\chi^b + \frac{1}{2}(\Gamma^{ab})_\beta{}^\alpha\chi_{abc} + \frac{1}{4}(\Gamma_c\Gamma_{abd})_\beta{}^\alpha\chi^{abd}$$

$$-\frac{1}{48}(\Gamma^{abde})_\beta{}^\alpha(\frac{1}{4}F^{(+)}_{abdec} + \chi^{(-)}_{abdec})] + \frac{1}{2}E^a E^b T_{ba}{}^{*\alpha}, \tag{2.12}$$

$$R^{ab} = 3(E^*\Gamma^{abc}E)\chi_c - 2(E^*\Gamma_c E)\chi^{abc} + \frac{1}{2}(E^*\Gamma^a\Gamma^{cde}\Gamma^b E)\chi_{cde} + \frac{1}{6}(E^*\Gamma_{cde}E)(\frac{1}{4}F^{(+)abcde} + \chi^{(-)abcde})$$

$$-\frac{1}{4}[(E\Gamma_c E)H^{*abc} + (E^*\Gamma_c E^*)H^{abc}] + \frac{1}{48}[(E\Gamma^a\Gamma^{cde}\Gamma^b E)H^*_{cde} + (E^*\Gamma^a\Gamma^{cde}\Gamma^b E^*)H_{cde}]$$

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$$+\frac{1}{2}E^c[(E^*\Theta_c^{ab}) + (E\Theta_c^{*ab})] + \frac{1}{2}E^c E^d R_{dc}{}^{ab}, \tag{2.13}$$

where

$$\Theta_{\alpha c}^{ab} = (\Gamma_c T^{ab})_\alpha - 2(\Gamma^{[a}T^{b]}{}_c)_\alpha,$$

$$\chi^{(r)} = \frac{1}{16}(\Lambda\Gamma^{(r)}\Lambda^*).$$

Notice that in our notation χ_a and χ_{abcde} are purely imaginary whereas χ_{abc} is real.

Moreover

$$\Delta V_+^i = V_-^i[-2(E\Lambda) + E^a R_a],$$

$$\Delta V_-^i = V_+^i[-2(E^*\Lambda^*) + E^a R_a^*], \tag{2.14}$$

$$H_3^i = +\frac{1}{2}[E^c(E\Gamma_c E)V_-^i + E^c(E^*\Gamma_c E^*)V_+^i] + \frac{1}{2}E^a E^b[(E^*\Gamma_{ba}\Lambda)V_-^i + (E\Gamma_{ba}\Lambda^*)V_+^i]$$

$$+\frac{1}{6}E^a E^b E^c[H_{abc}V_-^i + H_{abc}^*V_+^i], \tag{2.15}$$

$$F_5 = \frac{1}{6}E^a E^b E^c(E^*\Gamma_{cba}E) + \frac{1}{5!}E^a E^b E^c E^d E^e F_{abcde}, \tag{2.16}$$

and

$$\Delta\Lambda_\alpha = \frac{1}{2}R^a(\Gamma_a E^*)_\alpha + \frac{1}{24}(\Gamma^{abc}E)_\alpha H_{abc} + E^b\Delta_b\Lambda_\alpha,$$

$$\Delta\Lambda_\alpha^* = \frac{1}{2}R^{*a}(\Gamma_a E)_\alpha + \frac{1}{24}(\Gamma^{abc}E^*)_\alpha H_{abc}^* + E^b\Delta_b\Lambda_\alpha^*. \tag{2.17}$$

If $Z_{a_1\dots a_5}$ is a 5-indexed superfield,

$$Z^{(\pm)}{}_{a_1\dots a_5} = \frac{1}{2}(Z_{a_1\dots a_5} \pm (*Z)_{a_1\dots a_5}),$$

are its self-dual and antiself-dual components. $\chi_{a_1\dots a_5}$ is antiself-dual, i.e., $\chi_{a_1\dots a_5} = \chi^{(-)}_{a_1\dots a_5}$. Moreover

$$F^{(-)}_{a_1\dots a_5} = -8\chi_{a_1\dots a_5},$$

and

$$Z_{abcde} = \frac{1}{192}[F_{abcde} + 12\chi_{abcde}] = \frac{1}{192}[F_{abcde}^{(+)} + 4\chi_{abcde}^{(-)}].$$

From the definitions of R_1, R_1^* and Q one has

$$R_1 = -2(E\Lambda) + E^a R_a,$$

$$R_1^* = -2(E^*\Lambda^*) + E^a R_a^*. \tag{2.18}$$

3 Green-Schwarz Action, Rescaling and Field Redefinition

3.1 G-S Action

As a first step to obtain the pure spinor action, one must write the G-S action for the IIB sigma model. Since B_2^i is an $SL(2R)$ doublet and since the W-Z term involving B_2^i must be real and must have a scalar structure, we will introduce a complex, constant $SL(2R)$ doublet q_i (with $\bar{q}_i = (\tau_1 q^*)_i$) [30].

Notice that from the reality condition (2.1) and (2.2) one has $H_3^i = \bar{H}_3^i$ so that also $B_2^i = \bar{B}_2^i$. Therefore, writing $n^i = \frac{1}{2}(q_i + \bar{q}_i)$, one has that

$$\frac{1}{2}[q_i B_2^i + \bar{q}_i \bar{B}_2^i] = n_i B_2^i,$$

is real and "scalar". Moreover $n_i = \bar{n}_i$. Then by defining

$$e^{2\phi} = n_i V_-^i, \qquad e^{2\phi^*} = (n_i V_-^i)^* = \bar{n}_i V_+^i, \tag{3.1}$$

one has

$$n_i B_2^i = e^{2\phi} B_2 + e^{2\phi^*} B_2^*.$$

Now we propose the following G-S action (in the conformal gauge):

$$I_{GS} = \frac{1}{2} \int [e^\phi e^{\phi^*} E_+^a E_{-a} + 2n_i B_2^i]$$

$$= \int [\frac{1}{2} e^\phi e^{\phi^*} E_+^a E_{-a} + e^{2\phi} B_2 + e^{2\phi^*} B_2^*]. \tag{3.2}$$

The factor $e^\phi e^{\phi^*}$ in front of $E_+^a E_{-a}$ will become clear in the following.

3.2 Rescaling

The variation of the second term of I_{GS} involves the 3-superform $H_3 \equiv n_i H_3^i$ which, according to (2.15) and (3.1), is

$$H_3 = +\frac{1}{2}[E^c(E\Gamma_c E)e^{2\phi} + E^c(E^*\Gamma_c E^*)e^{2\phi^*}] + \frac{1}{2}E^a E^b[(E^*\Gamma_{ba}\Lambda)e^{2\phi} + (E\Gamma_{ba}\Lambda^*)e^{2\phi^*}]$$

$$+\frac{1}{6}E^a E^b E^c[H_{abc}e^{2\phi} + H_{abc}^* e^{2\phi^*}]. \tag{3.3}$$

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Eq. (3.3) suggests to perform the following rescaling

$$\epsilon^\alpha = \frac{e^\phi}{e^{k(\phi+\phi^*)}}E^\alpha, \qquad \epsilon^{*\alpha} = \frac{e^{\phi^*}}{e^{k(\phi+\phi^*)}}E^{*\alpha},$$

$$\mathcal{E}^a = e^{2k(\phi+\phi^*)}E^a,$$

$$\pi_\alpha = \frac{e^{3\phi}}{e^{(3k+1)(\phi+\phi^*)}}\Lambda_\alpha, \qquad \pi_\alpha^* = \frac{e^{3\phi^*}}{e^{(3k+1)(\phi+\phi^*)}}\Lambda_\alpha^*.$$

The simplest choice $k = 0$ yields unpleasant factors $\frac{1}{e^\phi e^{\phi^*}}$ in front of the torsions and curvatures expressed in terms of the rescaled fields. These unpleasant factors can be removed by choosing $k = \frac{1}{4}$. Then

$$\epsilon^\alpha = \frac{e^\phi}{e^{\frac{1}{4}(\phi+\phi^*)}}E^\alpha, \qquad \epsilon^{*\alpha} = \frac{e^{\phi^*}}{e^{\frac{1}{4}(\phi+\phi^*)}}E^{*\alpha}, \tag{3.4}$$

$$\mathcal{E}^a = e^{\frac{1}{2}(\phi+\phi^*)}E^a, \tag{3.5}$$

$$\pi_\alpha = \frac{e^{3\phi}}{e^{\frac{7}{4}(\phi+\phi^*)}}\Lambda_\alpha, \qquad \pi_\alpha^* = \frac{e^{3\phi^*}}{e^{\frac{7}{4}(\phi+\phi^*)}}\Lambda_\alpha^*, \tag{3.6}$$

and therefore

$$\kappa^{(r)} = \frac{1}{e^{\frac{1}{2}(\phi+\phi^*)}}\chi^{(r)} \equiv \frac{1}{16}(\pi\Gamma^{(r)}\pi^*), \tag{3.7}$$

$$\mathcal{H}_{abc} = \frac{e^{2\phi}}{e^{\frac{3}{2}(\phi+\phi^*)}}H_{abc}, \qquad \mathcal{H}_{abc}^* = \frac{e^{2\phi^*}}{e^{\frac{3}{2}(\phi+\phi^*)}}H_{abc}^*. \tag{3.8}$$

Moreover, from (2.14) and (2.16),

$$\rho^a = \frac{e^{4\phi}}{e^{\frac{5}{2}(\phi+\phi^*)}}R^a, \qquad \bar\rho^a = \frac{e^{4\phi^*}}{e^{\frac{5}{2}(\phi+\phi^*)}}\bar{R}^a, \tag{3.9}$$

and

$$\mathcal{F}_{abcde}^{(+)} = e^{\frac{5}{2}(\phi+\phi^*)}F_{abcde}^{(+)}. \tag{3.10}$$

Notice that B_2^i, C_4 and Ω^{ab} (and therefore H_3^i, F_5 and $R_a{}^b$) are not rescaled.

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3.3 Field redefinitions

Moreover from (2.14) one has

$$\Delta\phi = -(\epsilon^*\pi^*) + \frac{1}{2}\mathcal{E}^a\rho_a^*, \tag{3.11}$$

$$\Delta\phi^* = -(\epsilon\pi) + \frac{1}{2}\mathcal{E}^a\rho_a, \tag{3.12}$$

and the torsion of the rescaled vielbein $\mathcal{E}^a$ becomes

$$\Delta\mathcal{E}^a = (\epsilon^*\Gamma^a\epsilon) + [\frac{1}{2}(\epsilon\pi) + \frac{1}{2}(\epsilon^*\pi^*)]\mathcal{E}^a + \frac{1}{4}\mathcal{E}^a\mathcal{E}^b(\rho_b + \rho_b^*), \tag{3.13}$$

which does not vanish in the sectors (1,1) and (2,0) [3]. It is convenient to perform a redefinition of the spinor-like vielbeins and of the Lorentz connection so that $\Delta'\mathcal{E}^a$ vanishes in the sectors (1,1) and (2,0) (where now Δ' denotes the covariant differential given in terms of the redefined connection Ω'^{ab}). Indeed, calling $\mathcal{E}$ the redefined spinor-like vielbeins, let us consider the transformations

$$\mathcal{E}^\alpha = \epsilon^\alpha - \frac{1}{2}\mathcal{E}^b(\Gamma_b\pi^*)^\alpha, \qquad \mathcal{E}^{*\alpha} = \epsilon^{*\alpha} - \frac{1}{2}\mathcal{E}^b(\Gamma_b\pi)^\alpha, \tag{3.14}$$

$$\Omega'^{ab} = \Omega^{ab} + \delta\Omega^{ab}, \tag{3.15}$$

where

$$\delta\Omega^{ab} = \frac{1}{2}[(\mathcal{E}\Gamma^{ab}\pi) + (\mathcal{E}^*\Gamma^{ab}\pi^*)] + 4\mathcal{E}^c\kappa_c{}^{ab} - \frac{1}{2}\mathcal{E}^c\delta_c^{[a}(\rho^* + \rho)^{b]}, \tag{3.16}$$

and π, π^* and ρ, ρ^* are defined in (3.6), (3.9) and κ^{abc} is defined in (3.7) with $\Gamma^{(r)} = \Gamma^{abc}$. Then one can immediately verify that under the transformations (3.14), (3.15) and (3.16)

$$\Delta'\mathcal{E}^a = (\mathcal{E}^*\Gamma^a\mathcal{E}). \tag{3.17}$$

It is then straightforward to compute the torsions and curvatures with the rescaled and redefined fields and forms. However before doing that, let us introduce the associated Extended Free Differential Algebra with its related torsions and curvatures.

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[3]Once a supervielbein basis is fixed, any n-superform ψ_n can be decomposed as $\psi_n = \sum \psi_{(p,q)}$, $(p + q = n)$, where $\psi_{(p,q)}$ is the component of ψ_n proportional to p vector-like and q spinor-like vielbeins. Then $\psi_{(p,q)}$ is called the (p, q) sector of ψ_n.

4 Extended Free Differential Algebra

4.1 BRST differential, extended forms and ghosts

A convenient way to get the BRST transformations of fields and ghosts is to consider the Extended Differential Algebra that amounts to the following recipe:

a) Define the hatted quantities as

$$\hat{d} = d + s + \delta \equiv d + \tilde{s} + \tilde{\delta},$$ (4.1)

$$\hat{\mathcal{E}}^a = \mathcal{E}^a + \lambda^a,$$ (4.2)

$$\hat{\mathcal{E}}^\alpha = \mathcal{E}^\alpha + \lambda^\alpha,$$

$$\hat{\mathcal{E}}^{*\alpha} = \mathcal{E}^{*\alpha} + \lambda^{*\alpha},$$ (4.3)

$$\hat{\Omega}_{ab} = \hat{\mathcal{E}}^C \Omega'_C{}^{ab} + \psi^{ab} \equiv \Omega'^{ab} + \tilde{\psi}^{ab},$$ (4.4)

$$\hat{B}_2^i = \hat{\mathcal{E}}^A \hat{\mathcal{E}}^B B_{BA}^i + \sigma_1^i \equiv B^i + \tilde{\sigma}_1^i,$$ (4.5)

$$\hat{C}_4 = \hat{\mathcal{E}}^{A_1} \cdots \hat{\mathcal{E}}^{A_4} C_{A_1 \cdots A_4} + \sigma_3 \equiv C_4 + \tilde{\sigma}_3,$$ (4.6)

where the ghosts λ^a, λ^α, $\lambda^{*\alpha}$, ψ^{ab}, σ_1^i and σ_3 have ghost number $n_{gh} = 1$ but σ_1^i and σ_3, being one-form and 3-form respectively, contain ghosts of ghosts of higher ghost number. Since the Green-Schwarz action involves only the two-superform $n_i B_2^i$ for our purposes the relevant ghost related to B_2^i is $(n_i \sigma_1^i)$.

b) Assume that the ghost λ^a related to $\mathcal{E}^a$ and the ghost $n_i \tilde{\sigma}_1^i$ vanish so that

$$\hat{\mathcal{E}}^a = \mathcal{E}^a,$$ (4.7)

$$n_i \hat{B}_2^i = n_i B_2^i.$$ (4.8)

c) Write the extended parametrization for hatted torsions and curvatures simply copying that of the unhatted ones.

Following [15], one can give a geometric interpretation to Eqs. (4.1) - (4.6) by adding an odd, non-dynamical dimension to the superspace, with odd coordiante η . Then the ghosts (and ghosts of ghosts) in Eqs. (4.2) - (4.6) can be identified with the components of the corresponding hatted superforms along this odd direction and s in (4.1) as the differential along η.

A justification of the assumption expressed under the point b) can be done in the framework of the superembedding approach. In this approach the w.s. is considered a super w.s. and the ghosts which arise in the definitions of the hatted superforms are just the pull-back of these

extended superforms along an odd super w.s. direction, let say, of odd coordinate κ. Then the condition $\lambda_{ws}^a = d\kappa \partial_\kappa Z^M \mathcal{E}_M^a = 0$ is just the fundamental constraint of the superembedding approach, i.e., the requirement that the pull-back of the vector-like vielbeins along an odd super w.s. direction, vanishes. On this line, the condition $(n_i\tilde\sigma_{ws1}^i) = d\kappa \partial_\kappa Z^M \mathcal{E}_M^A \mathcal{E}^B B_{BA} = 0$, together with $\lambda_{ws}^a = 0$ expresses the fact that if, as in reonomic approach, one writes the G-S action as the integral of a top 2-form in the extended superspace, the pullback of this top 2-form vanishes along the odd super w.s. direction κ.

However, λ^a and λ_{ws}^a (as well as $(n_i\tilde\sigma_1^i)$ and $(n_i\tilde\sigma_{ws1}^i)$) are *a priori* different objects since λ^a and $(n_i\tilde\sigma_1^i)$ are the odd components of $\hat{\mathcal{E}}^a$ and $n_i\hat B_2^i$ in the extended superspace (Z^M, η) whereas λ_{ws}^a and $(n_i\tilde\sigma_{ws1}^i)$ are the odd components of the pull-back of $\hat{\mathcal{E}}^a$ and $n_i\hat B_2^i$ in the extended w.s. (ξ^i, κ). They can be identified if one specifies the odd part of the embedding of the extended w.s. on the extended superspace. Indeed, if $Z^M(\xi, \kappa) = Z^M(\xi) + \kappa Y^M(\xi)$ is this embedding, one can choose $\partial_\kappa Z^M \equiv Y^M = \mathcal{E}_\eta^A \mathcal{E}_A^M$ so that $\lambda^a = \lambda_{ws}^a$ and $n_i\tilde\sigma_1^i = n_i\tilde\sigma_{ws1}^i$ modulo the fact that $n_i\tilde\sigma_1^i$ is a full one-superform in the superspace and $n_i\tilde\sigma_{ws1}^i$ is its pull-back on the w.s.

Now we must be more precise about the action of δ. There are two equivalent options (δ and $\tilde\delta$):

1) δ induces Lorentz and gauge transformations with parameters ψ^{ab}, σ_1^i and σ_3. In this case δ and s anticommute and s is nilpotent. But s induces also Lorentz and gauge transformations with parameters $\lambda^\gamma \Omega'_\gamma{}^{ab} + \lambda^{*\gamma}\Omega'^{*ab}_\gamma$ etc.

2) $\tilde\delta$ induces Lorentz and gauge transformations with parameters $\tilde\psi^{ab}$, $\tilde\sigma_1$, and $\tilde\sigma_3$. Writing $s + \delta = \tilde s + \tilde\delta$, now $\tilde\delta$ and $\tilde s$ do not anticommute and $\tilde s$ is not nilpotent. Now $\tilde s$ induces covariant transformations ($\tilde s$ is the covariant BRST differential).

The BRST transformations of fields and ghost can be obtained by expanding in ghost number the parametrizations of the extended curvatures.

In the sector with ghost number $n_{gh} = 0$ it reproduces the parametrization on which we started. In the sector with $n_{gh} = 1$ it gives the BRST transformations of fields and forms. In the sector with $n_{gh} = 2$ it yields the transformations of the ghosts (and the ghost constraints as we shall see).

4.2   Parametrization of Extendended Torsions and Curvatures

As already noted, it is straightforward to compute the parametrization of torsions and curvatures with the rescaled and redefined fields and forms. Here we will give the results of this computation directly for the extended objects.

The extended version of (3.11) and (3.17) are

$$\hat\Delta\phi = -(\hat{\mathcal{E}}^*\pi^*) + \frac{1}{2}\mathcal{E}^a(\rho_a^* - 16\kappa_a), \tag{4.9}$$

$$\hat\Delta\mathcal{E}^a = (\hat{\mathcal{E}}^*\Gamma^a\hat{\mathcal{E}}), \tag{4.10}$$

where, in (4.10) and in the following, $\hat\Delta$ denotes the extension of Δ', the covariant differential associated to the Lorentz connection Ω'^{ab}, defined in (3.15) and (3.16).

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Then

$$\hat{\Delta}\pi_\alpha = -\frac{1}{24}\hat{\mathcal{E}}^\beta(\Gamma^{abc})_{\beta\alpha}(\mathcal{H}_{abc} - \frac{1}{4}\pi_{abc}) + \hat{\mathcal{E}}^{*\beta}[(\Gamma^c)_{\beta\alpha}(8\kappa_c + \frac{1}{2}\rho_c) - \frac{1}{3}(\Gamma^{abc})_{\beta\alpha}\kappa_{abc}] + \mathcal{E}^c\Delta_c\pi_\alpha \quad (4.11)$$

where $\pi_{abc} = (\pi\Gamma_{abc}\pi)$ (and $\pi^*_{abc} = (\pi^*\Gamma_{abc}\pi^*)$).

$$\hat{\Delta}\hat{\mathcal{E}}^\alpha = \frac{1}{2}[(\hat{\mathcal{E}}\Gamma^c\hat{\mathcal{E}})(\Gamma_c\pi)^\alpha + (\hat{\mathcal{E}}^*\Gamma^c\hat{\mathcal{E}}^*)(\Gamma_c\pi)^\alpha] + (\hat{\mathcal{E}}\Gamma^c\hat{\mathcal{E}}^*)(\Gamma_c\pi^*)^\alpha - \hat{\mathcal{E}}^\alpha[(\hat{\mathcal{E}}\pi) + (\hat{\mathcal{E}}^*\pi^*)]$$

$$-\hat{\mathcal{E}}^{*\alpha}[(\hat{\mathcal{E}}^*\pi) + (\hat{\mathcal{E}}\pi^*)] + \frac{1}{8}\mathcal{E}^c(\hat{\mathcal{E}}^*\Gamma^{ab})^\alpha[(\mathcal{H}_{abc} + \mathcal{H}^*_{abc}) - \frac{3}{4}(\pi_{abc} + \pi^*_{abc})]$$

$$+\mathcal{E}^c\hat{\mathcal{E}}^{*\beta}\frac{1}{48}(\Gamma^c\Gamma^{abd})_\beta{}^\alpha[(\mathcal{H}_{abd} - \mathcal{H}^*_{abd}) - \frac{1}{4}(\pi_{abd} - \pi^*_{abd})]$$

$$+\mathcal{E}^c\hat{\mathcal{E}}^\beta\{\frac{1}{48}(\Gamma^{abde})_\beta{}^\alpha\frac{1}{4}\mathcal{F}^{(+)}_{abdec} + (\Gamma_c\Gamma^b)^\alpha_\beta[4\kappa_b - \frac{1}{8}(\rho_b - \rho^*_b)]\} + \frac{1}{2}\mathcal{E}^c\mathcal{E}^b\tau^\alpha_{bc}, \quad (4.12)$$

$$(n_i\hat{H}^i_3) \equiv \hat{d}(n_i\hat{B}^i_2) = \frac{1}{2}\mathcal{E}^a[(\hat{\mathcal{E}}\Gamma_a\hat{\mathcal{E}}) + (\hat{\mathcal{E}}^*\Gamma_a\hat{\mathcal{E}}^*)] + \mathcal{E}^a\mathcal{E}^b\mathcal{E}^c[\frac{1}{6}(\mathcal{H}_{abc} + \mathcal{H}^*_{abc}) - \frac{1}{8}(\pi_{abc} + \pi^*_{abc})] \quad (4.13)$$

$$\hat{F}_5 = \frac{1}{6}\mathcal{E}^a\mathcal{E}^b\mathcal{E}^c(\hat{\mathcal{E}}^*\Gamma_{cba}\hat{\mathcal{E}}) - \frac{1}{12}\mathcal{E}^a\mathcal{E}^b\mathcal{E}^c\mathcal{E}^d[(\hat{\mathcal{E}}^*\Gamma_{abcd}\pi^*) + (\hat{\mathcal{E}}\Gamma_{abcd}\pi)] \quad (4.14)$$

$$+\frac{1}{5!}\mathcal{E}^a\mathcal{E}^b\mathcal{E}^c\mathcal{E}^d\mathcal{E}^e[\mathcal{F}^{(+)}_{abcde} - 3\kappa^{(-)}_{abcde}]. \quad (4.15)$$

Moreover, if we call $\hat{R}^{ab}$ the extended curvature of the redefined Lorentz connection Ω'^{ab} one has

$$\hat{R}^{ab} = \{-\frac{1}{4}[\hat{\mathcal{E}}\Gamma_f\hat{\mathcal{E}}) + (\hat{\mathcal{E}}^*\Gamma_f\hat{\mathcal{E}}^*)][(\mathcal{H}^{fab} + \mathcal{H}^{*fab}) - \frac{3}{4}(\pi^{fab} + \pi^{*fab})] - \frac{1}{48}[(\hat{\mathcal{E}}\Gamma^a\Gamma^{fgh}\Gamma^b\hat{\mathcal{E}})$$

$$-(\hat{\mathcal{E}}^*\Gamma^a\Gamma^{fgh}\Gamma^b\hat{\mathcal{E}}^*)][(\mathcal{H}_{fgh} - \mathcal{H}^*_{fgh}) - \frac{1}{4}(\pi_{fgh} - \pi^*_{fgh})] + \frac{1}{16\cdot 5!}(\hat{\mathcal{E}}\Gamma^a\Gamma_{fghlm}\Gamma^b\hat{\mathcal{E}}^*)\mathcal{F}^{(+)fghlm}$$

$$+(\hat{\mathcal{E}}\Gamma^a\Gamma^f\Gamma^b\hat{\mathcal{E}}^*)[4\kappa_f - \frac{1}{8}(\rho_f - \rho^*_f)] + c.c.\} - \frac{1}{2}\mathcal{E}^c[(\hat{\mathcal{E}}^*\vartheta^{ab}_c) + (\hat{\mathcal{E}}\vartheta^{*ab}_c)] + \frac{1}{2}\mathcal{E}^c\mathcal{E}^d\mathcal{R}_{dc}{}^{ab}, \quad (4.16)$$

where the explicit forms of ϑ^{ab}_c and $\mathcal{R}_{dc}{}^{ab}$ in terms of the other fields are not needed. Notice that $\hat{F}_5$ is purely imaginary while $\hat{R}^{ab}$, $\hat{H}_3$ and $\hat{\Delta}\hat{\mathcal{E}}^a$ are real, and the parametrizations of $\hat{\Delta}\phi^*$, $\hat{\Delta}\pi$ and $\hat{\Delta}\hat{\mathcal{E}}^{*\alpha}$ can be obtained by taking the complex conjugate of (4.9), (4.11) and (4.12), respectively.

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4.3   Pure spinor constraints

As already mentioned, equations (4.9) - (4.16) at ghost number $n_{gh} = 0$ give the parametrizations of the torsions and curvatures of the rescaled and redefined fields and forms and at ghost number $n_{gh} = 1$ they give the BRST transformations of these fields and forms. Now we are interested in the sector with ghost number $n_{gh} = 2$ where these equations give the BRST transformations of the ghosts. The vanishing of λ^a, i.e., Eq. (4.7) together with (4.10) at $n_{gh} = 2$ yields the constraint

$$(\lambda \Gamma^a \lambda^*) = 0, \tag{4.17}$$

and the vanishing of $n_i \tilde{\sigma}_1^i$, i.e., equation (4.8) together with (4.13) implies $\mathcal{E}^a[(\lambda \Gamma_a \lambda) + (\lambda^* \Gamma_a \lambda^*)] = 0$ so that

$$(\lambda \Gamma^a \lambda) + (\lambda^* \Gamma^a \lambda^*) = 0. \tag{4.18}$$

However, it is fair to recall, as remarked before, that equation (4.8) is stronger than the condition $n_i \tilde{\sigma}_{1ws}^i = 0$ that follows from the requirement that the pull-back of the G-S lagrangian vanishes along κ. Indeed $n_i \tilde{\sigma}_{1ws}^i = 0$ only implies that the pull-back $\mathcal{E}_\pm^a[(\lambda \Gamma_a \lambda) + (\lambda^* \Gamma_a \lambda^*)]$ vanishes.

   The constraints (4.17), (4.18) gain a more standard form if one writes $\lambda^\alpha = \frac{1}{\sqrt{2}}(\lambda_1^\alpha + i\lambda_2^\alpha)$. In fact, in terms of λ_1^α and λ_2^α these constraints become

$$(\lambda_1 \Gamma^a \lambda_1) = 0 = (\lambda_2 \Gamma^a \lambda_2), \tag{4.19}$$

i.e., the pure spinor constraints for λ_1 and λ_2. Moreover, (4.12) in the sector $n_{gh} = 2$ gives the BRST transformation of λ^α

$$\tilde{s}\lambda^\alpha = -\lambda^\alpha[(\lambda\pi) + (\lambda^*\pi^*)] - \lambda^{*\alpha}[(\lambda^*\pi) + (\lambda\pi^*)], \tag{4.20}$$

or in terms of λ_1 and λ_2,

$$\tilde{s}\lambda_1^\alpha = -2\lambda_1^\alpha(\lambda_1\pi_1), \qquad \tilde{s}\lambda_2^\alpha = 2\lambda_2^\alpha(\lambda_2\pi_2). \tag{4.21}$$

4.4   Real and Imaginary Components of Spinor-like Vielbeins and Other Fields

These results suggest that it should be convenient to rewrite equations (4.10), (4.12), (4.13), (4.15) and (4.16) in terms of the real and imaginary components of the relevant fields by writing

$$\mathcal{E}^\alpha = \frac{1}{\sqrt{2}}(\mathcal{E}_1^\alpha + i\mathcal{E}_2^\alpha),$$

$$\pi^\alpha = \frac{1}{\sqrt{2}}(\pi_1^\alpha + i\pi_2^\alpha), \tag{4.22}$$

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but we will define $\mathcal{H}_{abc} = \mathcal{H}_{1abc} + i\mathcal{H}_{2abc}$ and $\rho_a = \rho_{1a} + i\rho_{2a}$. Then equations (4.10) - (4.16) yield

$$\hat{\Delta}\mathcal{E}^a = \frac{1}{2}[(\hat{\mathcal{E}}_1\Gamma^a\hat{\mathcal{E}}_1) + (\hat{\mathcal{E}}_2\Gamma^a\hat{\mathcal{E}}_2)], \tag{4.23}$$

$$\hat{\Delta}\hat{\mathcal{E}}_1^\alpha = (\hat{\mathcal{E}}_1\Gamma^c\hat{\mathcal{E}}_1)(\Gamma_c\pi_1)^\alpha - 2\hat{\mathcal{E}}_1^\alpha(\hat{\mathcal{E}}_1\pi_1) + \frac{1}{4}\mathcal{E}^c\hat{\mathcal{E}}_1^\beta(\Gamma_{ab})_\beta{}^\alpha\tilde{\mathcal{H}}_{abc}$$

$$-\mathcal{E}^c(M\Gamma_c\hat{\mathcal{E}}_2)^\alpha + \frac{1}{2}\mathcal{E}^a\mathcal{E}^b\tau_{1,ba}^\alpha, \tag{4.24}$$

$$\hat{\Delta}\hat{\mathcal{E}}_2^\alpha = -[(\hat{\mathcal{E}}_2\Gamma^c\hat{\mathcal{E}}_2)(\Gamma_c\pi_2)^\alpha - 2\hat{\mathcal{E}}_2^\alpha(\hat{\mathcal{E}}_2\pi_2)] + \frac{1}{4}\mathcal{E}^c\hat{\mathcal{E}}_2^\beta(\Gamma_{ab})_\beta{}^\alpha\tilde{\mathcal{H}}_{abc}]$$

$$+\mathcal{E}^c(\hat{\mathcal{E}}_1\Gamma_cM)^\alpha + \frac{1}{2}\mathcal{E}^a\mathcal{E}^b\tau_{2,ba}^\alpha, \tag{4.25}$$

where we have defined

$$\tilde{\mathcal{H}}_{abc} = \mathcal{H}_{1abc} - \frac{3}{4}[(\pi_1\Gamma_{abc}\pi_1) - (\pi_2\Gamma_{abc}\pi_2)], \tag{4.26}$$

and

$$M^{\beta\alpha} = (\Gamma^b)^{\beta\alpha}(\frac{1}{4}\rho_{2b} + 4\bar{\kappa}^b) + \frac{1}{16\cdot 5!}(\Gamma^{abcde})^{\beta\alpha}\bar{\mathcal{F}}_{abcde}^{(+)}$$

$$+\frac{1}{24}(\Gamma^{bcd})^{\beta\alpha}[\mathcal{H}_{2bcd} - \frac{1}{4}(\pi_1\Gamma_{abc}\pi_2)], \tag{4.27}$$

and we have written $\bar{\kappa}_a = i\kappa_a$, $\bar{\kappa}_{abcde}^{(-)} = i\kappa_{abcde}^{(-)}$ and $\bar{\mathcal{F}}_{abcde}^{(+)} = i\mathcal{F}_{abcde}^{(+)}$ so that $\bar{\kappa}_a$, $\bar{\kappa}_{abcde}^{(-)}$ and $\bar{\mathcal{F}}_{abcde}^{(+)}$ are real. The expression of the fields $\tau_{1/2,ab}$ is irrelevant for our purposes. Moreover

$$(n_i\hat{H}_3^i) \equiv \hat{d}(n_i\hat{B}_2^i) = \frac{1}{2}\mathcal{E}^a[(\hat{\mathcal{E}}_1\Gamma_a\hat{\mathcal{E}}_1) - (\hat{\mathcal{E}}_2\Gamma_a\hat{\mathcal{E}}_2)] + \frac{1}{3}\mathcal{E}^a\mathcal{E}^b\mathcal{E}^c\tilde{\mathcal{H}}_{abc}, \tag{4.28}$$

$$\hat{F}_5 = \frac{i}{6}\{\mathcal{E}^a\mathcal{E}^b\mathcal{E}^c(\hat{\mathcal{E}}_1\Gamma_{cba}\hat{\mathcal{E}}_2) - \frac{1}{2}\mathcal{E}^a\mathcal{E}^b\mathcal{E}^c\mathcal{E}^d(\hat{\mathcal{E}}_2\Gamma_{abcd}\pi_2) + \frac{1}{20}\mathcal{E}^a\mathcal{E}^b\mathcal{E}^c\mathcal{E}^d\mathcal{E}^e[\bar{\mathcal{F}}_{abcde}^{(+)} - 3\bar{\kappa}_{abcde}^{(-)}]\}, \tag{4.29}$$

and

$$\hat{R}^{ab} = -\frac{1}{4}[(\hat{\mathcal{E}}_1\Gamma_f\hat{\mathcal{E}}_1) - (\hat{\mathcal{E}}_2\Gamma_f\hat{\mathcal{E}}_2)]\tilde{\mathcal{H}}^{fab} + 2(\hat{\mathcal{E}}_1\Gamma^aM\Gamma^b\hat{\mathcal{E}}_2) - \frac{1}{2}\mathcal{E}^c[(\hat{\mathcal{E}}_1\vartheta_{1c}^{ab}) + (\hat{\mathcal{E}}_2\vartheta_{2c}^{ab})]$$

$$+\frac{1}{2}\mathcal{E}^c\mathcal{E}^d\mathcal{R}_{dc}{}^{ab}. \tag{4.30}$$

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Using the identity

$$(\mathcal{E}_1 \Gamma^a \mathcal{E}_1)(\Gamma_a \pi_1)^\alpha - 2\mathcal{E}_1^\alpha(\mathcal{E}_1 \pi_1) = -\frac{1}{4}[(\Gamma^{ab}\mathcal{E}_1)^\alpha(\mathcal{E}_1 \Gamma_{ab} \pi_1) - 2\mathcal{E}_1^\alpha(\mathcal{E}_1 \pi_1)],$$

and a similar one for $\mathcal{E}_2$, the sectors (0,2) of $\hat{\Delta}\hat{\mathcal{E}}_i$ can be rewritten as

$$(\hat{\Delta}\hat{\mathcal{E}}_1^\alpha)_{(0,2)} = -\frac{1}{4}[(\Gamma^{ab}\hat{\mathcal{E}}_1)^\alpha(\hat{\mathcal{E}}_1 \Gamma_{ab} \pi_1) - 2\hat{\mathcal{E}}_1^\alpha(\hat{\mathcal{E}}_1 \pi_1)], \tag{4.31}$$

$$(\hat{\Delta}\hat{\mathcal{E}}_2^\alpha)_{(0,2)} = \frac{1}{4}[(\Gamma^{ab}\hat{\mathcal{E}}_2)^\alpha(\hat{\mathcal{E}}_2 \Gamma_{ab} \pi_2) - 2\hat{\mathcal{E}}_2^\alpha(\hat{\mathcal{E}}_2 \pi_2)]. \tag{4.32}$$

Moreover

$$\hat{\Delta}\pi_{1\alpha} = \hat{\mathcal{E}}_1^\beta[-\frac{1}{24}(\Gamma^{abc})_{\beta\alpha}\tilde{\mathcal{H}}_{abc} + \frac{1}{2}(\Gamma^c)_{\beta\alpha}\rho_{1c} - \frac{1}{48}(\Gamma^{abc})_{\beta\alpha}(\pi_1\Gamma_{abc}\pi_1)]$$

$$-\frac{1}{4}(\Gamma^c M \Gamma_c \hat{\mathcal{E}}_2)_\alpha + \mathcal{E}^c\hat{\Delta}\pi_{1\alpha}, \tag{4.33}$$

$$\hat{\Delta}\pi_{2\alpha} = \hat{\mathcal{E}}_2^\beta[-\frac{1}{24}(\Gamma^{abc})_{\beta\alpha}\tilde{\mathcal{H}}_{abc} + \frac{1}{2}(\Gamma^c)_{\beta\alpha}\rho_{1c} - \frac{1}{48}(\Gamma^{abc})_{\beta\alpha}(\pi_2\Gamma_{abc}\pi_2)]$$

$$+\frac{1}{4}(\hat{\mathcal{E}}_1\Gamma^c M \Gamma_c)_\alpha + \mathcal{E}^c\hat{\Delta}\pi_{2\alpha}. \tag{4.34}$$

The parametrizations of torsions and curvatures are obtained by looking at the sector with $n_{gh} = 0$ of equations (4.22) - (4.34), i.e., dropping the hats in these equations.

As for the sector with $n_{gh} = 1$, let us report only the BRST transformations of the super-vielbeins $\mathcal{E}^a$, $\mathcal{E}_i^\alpha$ and the B-fields $n_i B_2^i$:

$$\tilde{s}\mathcal{E}^a = (\lambda_1 \Gamma^a \mathcal{E}_1) + (\lambda_2 \Gamma^a \mathcal{E}_2), \tag{4.35}$$

$$\tilde{s}\mathcal{E}_1^\alpha = -\Delta\lambda_1^\alpha - \frac{1}{4}(\Gamma^{ab}\lambda_1)^\alpha(\mathcal{E}_1 \Gamma_{ab} \pi_1) + \frac{1}{2}\lambda_1^\alpha(\mathcal{E}_1 \pi_1) - \frac{1}{4}\mathcal{E}^c(\Gamma_{ab}\lambda_1)^\alpha\tilde{\mathcal{H}}_{abc}$$

$$-\frac{1}{4}(\Gamma^{ab}\mathcal{E}_1)^\alpha(\lambda_1 \Gamma_{ab} \pi_1) + \frac{1}{2}\mathcal{E}_1^\alpha(\lambda_1 \pi_1) - \mathcal{E}^c(M\Gamma_c\lambda_2)^\alpha, \tag{4.36}$$

$$\tilde{s}\mathcal{E}_2^\alpha = -\Delta\lambda_2^\alpha + \frac{1}{4}(\Gamma^{ab}\lambda_2)^\alpha(\mathcal{E}_2 \Gamma_{ab} \pi_2) - \frac{1}{2}\lambda_2^\alpha(\mathcal{E}_2 \pi_2)] + \frac{1}{4}\mathcal{E}^c(\Gamma_{ab}\lambda_2)^\alpha\tilde{\mathcal{H}}_{abc}$$

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$$+\frac{1}{4}(\Gamma^{ab}\mathcal{E}_2)^{\alpha}(\lambda_2\Gamma_{ab}\pi_2) - \frac{1}{2}\mathcal{E}_2^{\alpha}(\lambda_2\pi_2)] + \mathcal{E}^c(\lambda_1\Gamma_c M)^{\alpha}, \tag{4.37}$$

$$s(n_i B_2^i) = \mathcal{E}^a[(\lambda_1\Gamma_a\mathcal{E}_1) - (\lambda_2\Gamma_a\mathcal{E}_2)]. \tag{4.38}$$

In this notation, the Green-Schwarz action (3.2) is

$$I_{GS} = \frac{1}{2}\int[\mathcal{E}_+^a\mathcal{E}_{-a} + 2n_i B_2^i], \tag{4.39}$$

where $\mathcal{E}_{\pm}^A$ are the left-handed and right-handed pullbacks of the supervielbeins on the world-sheet. The BRST transformation of I_{GS} is

$$sI_{GS} = \int[(\lambda_1\Gamma_a\mathcal{E}_+^a\mathcal{E}_{-1}) + (\lambda_2\Gamma_a\mathcal{E}_-^a\mathcal{E}_{+2})]. \tag{4.40}$$

Notice that, if one chooses $\lambda_1^{\alpha} = (k_1\Gamma^b\mathcal{E}_{+b})^{\alpha}$ and $\lambda_2^{\alpha} = (k_2\Gamma^b\mathcal{E}_{-b})^{\alpha}$ where k_i are local parameters, I_{GS} is invariant (modulo the Virasoro constraints). This is the κ-symmetry of the G-S action.

A useful identity that follows from the Bianchi identity $\hat{\Delta}\hat{R}^{ab} = 0$ in the sector with ghost number 3, is

$$(\lambda_1\Gamma^{[a}[\lambda_1^{\alpha}\Delta_{1\alpha}P + \lambda_2^{\alpha}\Delta_{2\alpha}P]\Gamma^{b]}\lambda_2) = 0, \tag{4.41}$$

where

$$M = e^{-2(\phi+\phi^*)}P.$$

If one defines

$$\lambda_1{}^{\alpha}\Delta_{1\alpha}P^{\beta\gamma} = \lambda_1{}^{\alpha}C_{1\alpha}{}^{\beta\gamma},$$

$$\lambda_2{}^{\alpha}\Delta_{2\alpha}P^{\beta\gamma} = \lambda_2{}^{\alpha}C_{2\alpha}{}^{\beta\gamma}, \tag{4.42}$$

(4.41) implies that $C_{1\alpha}{}^{\beta\gamma}$ and $C_{2\alpha}{}^{\beta\gamma}$ are Lorentz-Weyl valued in α, β and α, γ respectively.

It is interesting to note that the parametrization of torsions and curvatures, when expressed in terms of the real and imaginary parts of the rescaled and redefined fields and forms, has a structure very similar to that of IIA superstring [23], a non-surprising result given that it is also present from the beginning in the treatment of Berkovits and Howe in [11].

5 Pure Spinor Action

5.1 Antighosts, $d_{\underline{\alpha}}$ Fields and BRST charge

In order to derive the pure spinor action one must add to the superspace coordinates $Z^M = (X^a, \theta^{\mu})$ the ghosts

$$\lambda^{\underline{\alpha}} = (\lambda_1^{\alpha}, \lambda_2^{\alpha}),$$

the antighosts

$$\omega_{\underline{\alpha}} = (\omega_{1\alpha}, \omega_{2\alpha}),$$

with ghost number $n_{gh} = -1$ which will play the role of the conjugate momenta of $\lambda^{\underline{\alpha}}$, and the fields

$$d_{\underline{\alpha}} = (d_{1\alpha}, d_{2\alpha}),$$

that will also play the role of the conjugate momenta of θ^μ and are essentially the BRST partners of $\omega_{\underline{\alpha}}$. From the worldsheet point of view, ω_1, d_1 and ω_2, d_2 are left-handed and right-handed chiral fields, respectively.

An index $\underline{\alpha}$ repeated, like for instance in $\lambda^{\underline{\alpha}} d_{\underline{\alpha}}$, means $\lambda^{\underline{\alpha}} d_{\underline{\alpha}} = (\lambda_1 d_1) + (\lambda_2 d_2)$ whereas indices $i(i = 1, 2)$ repeated do not imply summation.

Since, as a consequence of the pure spinor constraints, $\lambda^{\underline{\alpha}}$ contains $11 + 11$ degrees of freedom, also $\omega_{\underline{\alpha}}$ should contain $11 + 11$ independent components. This is realized by assuming that the pure spinor action is invariant under the ω-gauge symmetry

$$\delta^{(\omega)}\omega_i = \Lambda_i^a(\Gamma_a \lambda_i) \qquad i = 1, 2, \tag{5.1}$$

where Λ_i^a are local gauge parameters. The $d_{\underline{\alpha}}$ allow us to define the BRST charge

$$Q = \oint (\lambda^{\underline{\alpha}} d_{\underline{\alpha}}) = \oint (\lambda_1 d_1) + \oint (\lambda_2 d_2), \tag{5.2}$$

that generates the transformations induced by the BRST differential s. It is also useful to split s as $s = s_1 + s_2$ where s_1 is generated by the charge $Q_1 = \oint(\lambda_1 d_1)$ and s_2 is generated by the charge $Q_2 = \oint(\lambda_2 d_2)$.

In order to specify Q, prove its nilpotence and compute the BRST transformations of λ_i, ω_i and d_i one needs the expression of $(\lambda_i d_i)$ which is expected to be

$$\lambda_1^\alpha d_{1\alpha} = \lambda_1^\alpha [d_{1\alpha}^{(0)} + (\Omega'_{\alpha\beta}{}^\gamma + X_{\alpha\beta}^{(1)\gamma})\omega_{1\gamma}\lambda_1^\beta + \Omega'_{\alpha\beta}{}^\gamma \omega_{2\gamma}\lambda_2^\beta], \tag{5.3}$$

$$\lambda_2^\alpha d_{2\alpha} = \lambda_2^\alpha [d_{2\alpha}^{(0)} + (\Omega'_{\alpha\beta}{}^\gamma + X_{\alpha\beta}^{(2)\gamma})\omega_{2\gamma}\lambda_2^\beta + \Omega'_{\alpha\beta}{}^\gamma \omega_{1\gamma}\lambda_1^\beta], \tag{5.4}$$

where $d_{\underline{\alpha}}^{(0)}$, acting on superfields, induces the tangent space derivative $D_{\underline{\alpha}}$ and $\Omega'_{\underline{\alpha}}{}^{\underline{\beta}}$ are the spinorial partners of the Lorentz connection defined in (3.15) and (3.16). The superfields $X^{(1)}$ and $X^{(2)}$ are needed to assure the nilpotence of Q and to reproduce equation (4.21).

As we will see, these two requirements are satisfied if one chooses for $X^{(i)}$

$$X_{\alpha\beta}^{(1)\gamma} = -\frac{1}{4}(\Gamma^{ab}\pi_1)_\alpha (\Gamma_{ab})_\beta{}^\gamma - \frac{1}{2}\pi_{1\alpha}\delta_\beta{}^\gamma, \tag{5.5}$$

$$X_{\alpha\beta}^{(2)\gamma} = \frac{1}{4}(\Gamma^{ab}\pi_2)_\alpha (\Gamma_{ab})_\beta{}^\gamma + \frac{1}{2}\pi_{2\alpha}\delta_\beta{}^\gamma]. \tag{5.6}$$

We shall write

$$X_{\underline{\alpha\beta}}{}^{\underline{\gamma}} = (X_{\alpha\beta}^{(1)\gamma}, X_{\alpha\beta}^{(2)\gamma}),$$

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and also define the Lorentz-Weyl connections $\tilde{\Omega}_{\underline{\beta}}{}^{\gamma} = \Omega'_{\underline{\beta}}{}^{\gamma} + X_{\underline{\beta}}{}^{\gamma}$ with being

$$X_{\underline{\beta}}{}^{\gamma} = \mathcal{E}^{\underline{\alpha}} X_{\underline{\alpha}\underline{\beta}}{}^{\gamma} + \mathcal{E}^{c} \mathcal{H}_{c\underline{\beta}}{}^{\gamma}.$$

It is convenient to choose

$$\mathcal{H}_{c\underline{\beta}}{}^{\gamma} = \frac{1}{4}(\Gamma^{ab})_{\underline{\beta}}{}^{\gamma} \tilde{\mathcal{H}}_{abc},$$

so that, calling $\tilde{\Delta}$ the covariant differential related to the Lorentz-Weyl connection $\tilde{\Omega}$, the torsion $\hat{\tilde{T}}^{\underline{\alpha}} \equiv \hat{\tilde{\Delta}} \hat{\mathcal{E}}^{\underline{\alpha}}$ vanishes in the sector (0,2), and in the sector (1, 1) it becomes

$$(\hat{\tilde{T}}^{\underline{\alpha}})_{(1,1)} = (-\mathcal{E}^{c}(M\Gamma_{c}\hat{\mathcal{E}}_{2})^{\alpha}, \quad \mathcal{E}^{c}(\hat{\mathcal{E}}_{1}\Gamma_{c}M)^{\hat{\alpha}}). \tag{5.7}$$

Then, from the Bianchi identity $\tilde{\Delta}\tilde{\Delta}\mathcal{E}^{\underline{\alpha}} = \mathcal{E}^{\underline{\beta}}\tilde{R}_{\underline{\beta}}{}^{\underline{\alpha}}$ one obtains for $\mathcal{E}^{\underline{\beta}}\tilde{R}_{\underline{\beta}}{}^{\underline{\alpha}}$ the simple result

$$(\mathcal{E}_{1}\tilde{R}^{(1)\gamma})_{(0,3)} = -\frac{1}{2}(\mathcal{E}_{1}\Gamma^{a}\mathcal{E}_{1})(M\Gamma_{a}\mathcal{E}_{2})^{\gamma}, \tag{5.8}$$

$$(\mathcal{E}_{2\beta}\tilde{R}_{\beta}^{(2)\gamma})_{(0,3)} = \frac{1}{2}(\mathcal{E}_{2}\Gamma^{a}\mathcal{E}_{2})(\mathcal{E}_{1}\Gamma_{a}M)^{\gamma}. \tag{5.9}$$

Now, in this notation,

$$d_{\underline{\alpha}} = d_{\underline{\alpha}}^{(0)} + \tilde{\Omega}_{\underline{\alpha}\underline{\beta}}{}^{\gamma} \lambda^{\underline{\beta}} \omega_{\gamma}, \tag{5.10}$$

and

$$Q = \oint \lambda^{\underline{\alpha}}(d_{\underline{\alpha}}^{(0)} + \tilde{\Omega}_{\underline{\alpha}\underline{\beta}}{}^{\gamma} \lambda^{\underline{\beta}} \omega_{\gamma}). \tag{5.11}$$

Moreover the BRST transformations of λ_{i} and ω_{i} become

$$s\lambda^{\underline{\alpha}} = \lambda^{\underline{\beta}} \lambda^{\underline{\gamma}} \tilde{\Omega}_{\underline{\beta}\gamma}{}^{\underline{\alpha}}, \tag{5.12}$$

and

$$s\omega_{\underline{\alpha}} = -d_{\underline{\alpha}} - \lambda^{\underline{\beta}} \tilde{\Omega}_{\underline{\beta}\underline{\alpha}}{}^{\gamma} \omega_{\underline{\gamma}}. \tag{5.13}$$

It is useful to notice that if one defines

$$Y_{\alpha\beta}^{(1)\gamma} = X_{\alpha\beta}^{(1)\gamma} + 2\pi_{1\alpha}\delta_{\beta}{}^{\gamma}, \qquad Y_{\alpha\beta}^{(2)\gamma} = X_{\alpha\beta}^{(2)\gamma} - 2\pi_{2\alpha}\delta_{\beta}{}^{\gamma}, \tag{5.14}$$

$Y_{\alpha\beta}^{(i)\gamma}$ are symmetric in α and β and therefore Lorentz-Weyl valued both in α, γ and in β, γ. Given (5.5) and (5.6), the covariant form of (5.12), that is, $\tilde{s}\lambda_{i}^{\alpha} = \lambda_{i}^{\beta} \lambda_{i}^{\gamma} X_{\beta\gamma}^{(i)\alpha}$ reproduces (4.21) and using (5.14)

$$\tilde{s}\omega_{i\alpha} = -d_{i\alpha} - \lambda_{i}^{\beta} Y_{\alpha\beta}^{(i)\gamma} \omega_{i\gamma} \mp 2(\lambda_{i}\pi_{i})\omega_{i\alpha}. \tag{5.15}$$

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To compute $sd_{\underline{\alpha}}$ we assume

$$\{\oint \lambda^{\underline{\beta}} d_{\underline{\beta}}^{(0)}, d_{\underline{\alpha}}^{(0)}\} = -(\Gamma^a \lambda)_{\underline{\alpha}}[\mathcal{E}_{\pm a} + \tilde{\Omega}_a], \tag{5.16}$$

where

$$\mathcal{E}_{\pm a} \lambda^{\underline{\beta}} \Gamma^a_{\underline{\beta\alpha}} = (\mathcal{E}^a_+ (\lambda_1 \Gamma_a)_\alpha, \quad \mathcal{E}^a_- (\lambda_2 \Gamma_a)_\alpha),$$

Then

$$sd_{\underline{\alpha}} = -(\Gamma^a \lambda)_{\underline{\alpha}} \mathcal{E}_{\pm a} + \lambda^{\underline{\beta}} \tilde{\Omega}_{\underline{\alpha\beta}}{}^{\underline{\gamma}} d_{\underline{\gamma}} + \lambda^{\underline{\delta}} \lambda^{\underline{\beta}} \tilde{R}_{\underline{\delta\alpha\beta}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}}, \tag{5.17}$$

so that

$$Q^2 = \oint \lambda^{\underline{\alpha}} \lambda^{\underline{\beta}} \lambda^{\underline{\delta}} \tilde{R}_{\underline{\alpha\beta\delta}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}}. \tag{5.18}$$

Then, it follows from (5.8) and (5.9) that indeed $Q^2 = 0$.

¿From (5.12) (or (4.21)) it also follows that

$$s^2 \lambda_{\underline{\alpha}} = 0,$$

in agreement with the nilpotence of Q.

However $s^2 \omega_{\underline{\alpha}}$ does not vanish since from (5.13) and (5.17) one has [4]

$$s^2 \omega_{\underline{\alpha}} = (\Gamma_a \lambda)_{\underline{\alpha}} \mathcal{E}_{\pm a} - 2\lambda^{\underline{\delta}} \lambda^{\underline{\beta}} \tilde{R}_{\underline{\delta(\alpha\beta)}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}},$$

that is, using (5.8) and (5.9),

$$s^2 \omega_{1\alpha} = (\Gamma_a \lambda_1)_\alpha [\mathcal{E}^a_+ + (\omega_1 M \Gamma_a \lambda_2)], \tag{5.19}$$

$$s^2 \omega_{2\alpha} = (\Gamma_a \lambda_2)_{\hat{\alpha}} [\mathcal{E}^a_- - (\lambda_1 \Gamma_a M \omega_2)]. \tag{5.20}$$

Also $s^2 d_{\underline{\alpha}}$ does not vanish since

$$s^2 d_{1\alpha} = -[(\mathcal{E}_{2+} \Gamma_a \lambda_2)(\lambda_1 \Gamma^a)_\alpha + s[(\omega_1 M \Gamma_a \lambda_2)(\lambda_1 \Gamma^a)_\alpha], \tag{5.21}$$

$$s^2 d_{2\alpha} = -[(\Gamma^a \lambda_2)_\alpha (\lambda_1 \Gamma_a \mathcal{E}_{1-}) - s[(\Gamma^a \lambda_2)_\alpha (\lambda_1 \Gamma_a M \omega_2)]. \tag{5.22}$$

The nonvanishing of (5.21) and (5.22) is not a problem since, as we will see later, the RHS of (5.21) and (5.22), vanishes on shell, being proportional to the fields equations of $d_{\underline{\alpha}}$

$$(\lambda_1 \Gamma^a \mathcal{E}_{1-}) - s(\lambda_1 \Gamma^a M \omega_2) = (\lambda_1 \Gamma^a)_\beta [\mathcal{E}^\beta_{1-} + (M d_2)^\beta - \tilde{C}_{2\alpha}{}^{\beta\gamma} \lambda_2^\alpha \omega_{2\gamma}] = 0, \tag{5.23}$$

$$(\lambda_2 \Gamma^a \mathcal{E}_{2+}) + s(\lambda_2 \Gamma^a M \omega_1) = (\lambda_2 \Gamma^a)_\beta [\mathcal{E}^\beta_{2+} - (d_1 M)^\beta + \tilde{C}_{1\alpha}{}^{\beta\gamma} \lambda_1^\alpha \omega_{1\gamma}] = 0, \tag{5.24}$$

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[4]In our notations $X_{(\alpha\beta)} = \frac{1}{2}(X_{\alpha\beta} + X_{\beta\alpha})$ and $X_{[\alpha\beta]} = \frac{1}{2}(X_{\alpha\beta} - X_{\beta\alpha})$.

where

$$\tilde{C}_{i\alpha}{}^{\beta\gamma} = C_{i\alpha}{}^{\beta\gamma} + Y_{i\alpha}{}^{\beta\gamma}, \tag{5.25}$$

and $C_{i\alpha}{}^{\beta\gamma}$ and $Y_{i\alpha}{}^{\beta\gamma}$ are defined in (4.42) and (5.14).

The failure of nilpotency in equations (5.19) and (5.20) is a consequence of the ω-gauge transformation (5.1). Indeed s^2, acting on ω, vanishes only modulo this gauge transformation.

One can cure this inconvenience by fixing the ω-gauge and a useful way to do that is to apply the so-called Y-formalism.

Given the constant spinors $V_{\underline{\alpha}} = (V_{1\alpha}, V_{2\alpha})$ one defines

$$K_{\underline{\alpha}}{}^{\underline{\beta}} = (K_\alpha^{(1)\beta}, \quad K_\alpha^{(2)\beta}),$$

where

$$K_\alpha^{(1)\beta} = \frac{1}{2}(\Gamma^a\lambda_1)_\alpha(Y_1\Gamma_a)^\beta,$$

$$K_\alpha^{(2)\beta} = \frac{1}{2}(\Gamma^a\lambda_2)_\alpha(Y_2\Gamma_a)^\beta, \tag{5.26}$$

and

$$Y_i = \frac{V_i}{(V_i\lambda_i)},$$

so that

$$(Y_i\lambda_i) = 1.$$

Moreover

$$(\lambda_1 K^{(1)})^\alpha = 0 = (\lambda_2 K^{(2)})^\alpha, \tag{5.27}$$

and

$$((1 - K^{(1)})\Gamma^a\lambda_1)_\alpha = 0 = ((1 - K^{(2)})\Gamma^a\lambda_2)_\alpha. \tag{5.28}$$

Here $K^{(1)}$ and $K^{(2)}$ are projectors and, since $TrK_i = 5$, they project on five-dimensional subspaces of the 16-dimensional spinorial spaces so that from (5.17) one can see that λ_1 and λ_2 have 11 independent components.

Using the projectors $K^{(i)}$ one can fix the ω-gauge symmetry by requiring

$$(K^{(1)}\omega_1)_\alpha = 0 = (K^{(2)}\omega_2)_\alpha, \tag{5.29}$$

or equivalently,

$$\omega_i = ((1 - K^{(i)})\omega_i), \tag{5.30}$$

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so that each of the ω_i also has 11 components. Moreover one can also split the fields $d_{\underline{\alpha}}$ as

$$d_i^{(\top)} = ((1 - K^{(i)})d_i,$$

$$d_i^{(\perp)} = K^{(i)}d_i.$$

Notice that only $d_i^{(\top)}$ appear in the BRST charge Q so that $d_i^{(\top)}$ are the BRST partners of ω_i. Since V_i are constants, $K^{(i)}$ break Lorentz invariance, and are singular at $(V_i\lambda_i) = 0$ but these facts are not a problem since, as we will see, any dependence on $K^{(i)}$ disappears in the final result.

Projecting (5.13), (5.17), with $(1 - K)$ one gets $s\omega_i$ and $sd_i^{(\top)}$ which, in covariant form are

$$\tilde{s}\omega_{\underline{\alpha}} = -d_{\underline{\alpha}}^{(\top)} - \lambda^{\underline{\beta}} X_{\underline{\beta\alpha}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}}, \tag{5.31}$$

$$\tilde{s}d_{\underline{\alpha}}^{(\top)} = \lambda^{\underline{\beta}} X_{\underline{\beta\alpha}}{}^{\underline{\gamma}} d_{\underline{\gamma}}^{(\top)} + \lambda^{\underline{\delta}}\lambda^{\underline{\beta}} \tilde{R}_{\underline{\alpha\delta\beta}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}}, \tag{5.32}$$

and projecting (5.19), (5.20), (5.8) and (5.9) with $(1 - K)$ one has

$$s^2\omega_{\underline{\alpha}} = 0 = s^2 d_{\underline{\alpha}}^{(\top)}. \tag{5.33}$$

As for $sd^{\perp}$, notice that, given the definition (5.10) of $d_{\underline{\alpha}}$, only the components of $\tilde{\Omega}_{\underline{\alpha\beta}}{}^{\underline{\gamma}}$ projected with $(1 - K)_{\underline{\beta'}}^{\underline{\beta}}$ and $(1 - K)_{\underline{\gamma}}^{\underline{\gamma'}}$ are present in (5.17) so that

$$\tilde{s}d_{1\alpha}^{(\perp)} = -(\Gamma_a\lambda_1)_\alpha[\mathcal{E}_+^a + (\omega_1 M\Gamma^a\lambda_2)], \tag{5.34}$$

$$\tilde{s}d_{2\alpha}^{(\perp)} = -(\Gamma_a\lambda_2)_\alpha[\mathcal{E}_-^a - (\lambda_1\Gamma^a M\omega_2)]. \tag{5.35}$$

Moreover $s^2 d_{\underline{\alpha}}^{\perp}$ is just given by (5.21) and (5.22) which, as we will see, vanishes on shell. Now s is nilpotent acting on any field or ghost.

5.2 Derivation of the Action

In [23] two methods are presented to derive the pure spinor action in IIA superstring σ-models. Both methods can be applied to the present case. In this subsection we will give the details of only the second method first proposed in [13] for heterotic σ-models.

The strategy is the following: as a first step one adds to the Green-Schwarz action I_{GS} a new action I_K that depends on the projector $K_{\underline{\alpha}}{}^{\underline{\beta}}$ such that the action $I_{GS} + I_K$ is BRST invariant; then one adds a BRST exact action term I_{gf} (the "gauge fixing" action), given by the BRST transformation of a "gauge fermion" with ghost number $n_{gh} = -1$ such that the BRST invariant action $I = I_{GS} + I_K + I_{gf}$ becomes independent of $K_{\underline{\alpha}}{}^{\underline{\beta}}$.

The Green-Schwarz action is given in (4.39) and its BRST transformation in (4.40). Now consider the action

$$I_K = I_K^{(1)} + I_K^{(2)} + I_K^{(3)},$$

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where

$$I_K^{(1)} = \oint [(\mathcal{E}_{1-} K^{(1)} d_1) + (\mathcal{E}_{2+} K^{(2)} d_2)], \tag{5.36}$$

$$I_K^{(2)} = -\oint (d_1 \tilde{K}^{(1)} M K^{(2)} d_2), \tag{5.37}$$

$$I_K^{(3)} = \oint (\omega_1 M \Gamma_a \lambda_2)(\lambda_1 \Gamma^a M \omega_2). \tag{5.38}$$

with $\tilde{K}^{(i)}$ being the transpose of $K^{(i)}$. Notice that equations (4.36) and (4.37), projected with K reduce to

$$\tilde{s}(\mathcal{E}_1 K^{(1)}))^\alpha = -\mathcal{E}^c (M \Gamma_c \lambda_2)^\alpha, \tag{5.39}$$

$$\tilde{s}(\mathcal{E}_2 K^{(2)})^\alpha = \mathcal{E}^c (\lambda_1 \Gamma_c M)^\alpha. \tag{5.40}$$

An explicit computation of sI_K, using equations (5.34), (5.35), (5.39) and (5.40) as well as (4.21) and (5.31) and taking into account equations (4.41) and (4.42) yields

$$sI_K = -\oint [(\lambda_1 \mathcal{E}_+^a \Gamma_a \mathcal{E}_{1-}) + (\lambda_2 \mathcal{E}_-^a \Gamma_a \mathcal{E}_{2+})]$$

$$- \oint (\omega_1 M \Gamma^a \lambda_2)(\lambda_1 \Gamma_a)_\beta [\mathcal{E}_{1-}^\beta + (M d_2)^\beta - C_{2\alpha}{}^{\beta\gamma} \lambda_2^\alpha \omega_{2\gamma}]$$

$$+ \oint [\mathcal{E}_2^\beta - (d_1 M)^\beta + C_{1\alpha}{}^{\beta\gamma} \lambda_1^\alpha \omega_{1\gamma}](\Gamma^a \lambda_2)_\beta (\lambda_1 \Gamma_a M \omega_2), \tag{5.41}$$

where the last two integrals vanish on shell, as discussed before (see (5.23), (5.24)) and proved later on. Therefore, given (4.40),

$$sI_{GS} + I_K = 0,$$

on shell.

Now we define I_{gf} as

$$I_{gf} = -s \oint [(\mathcal{E}_{1-} \omega_1) + (\mathcal{E}_{2+} \omega_2)] + s \oint [(d_1 \tilde{K}^{(1)} M \omega_2) - (\omega_1 M K^{(2)} d_2)]$$

$$-\frac{1}{2} s \oint (s_1 - s_2)(\omega_1 M \omega_2). \tag{5.42}$$

Performing the BRST variations one has

$$-s \oint [(\mathcal{E}_{1-} \omega_1) + \mathcal{E}_{2+} \omega_2)] = (\omega_1 \tilde{\Delta}_- \lambda_1) + (\omega_2 \tilde{\Delta}_+ \lambda_2) + \mathcal{E}_-^a (\omega_1 M \Gamma_a \lambda_2) - \mathcal{E}_+^a (\lambda_1 \Gamma_a M \omega_2)$$

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$$+\mathcal{E}_{1-}((1-K^{(1)})d_1)+\mathcal{E}_{2+}((1-K^{(2)})d_2), \tag{5.43}$$

$$+s\oint[(d_1\tilde{K}^{(1)}M\omega_2)-(\omega_1MK^{(2)}d_2)]=\mathcal{E}_+^a((\lambda_1\Gamma_aM\omega_2)-\mathcal{E}_-^a(\omega_1M\Gamma_a\lambda_2)$$

$$+(d_1K^{(1)})_\alpha\tilde{C}_{2\beta}^{\alpha\gamma}\lambda_2^\beta\omega_{2\gamma}+\omega_{1\gamma}\lambda_{1\beta}\tilde{C}_{1\beta}^{\gamma\alpha}(K^{(2)}d_2)_\alpha-(d_1K^{(1)}M(1-K^{(2)})d_2)$$

$$-(d_1(1-K^{(1)})MK^{(2)}d_2)-2(\omega_1M\Gamma_a\lambda_2)(\lambda_1\Gamma^aM\omega_2), \tag{5.44}$$

$$-\frac{1}{2}s\oint(s_1-s_2)(\omega_1M\omega_2)=(d_1(1-K^{(1)}))_\alpha\tilde{C}_{2\beta}^{\alpha\gamma}\lambda_2^\beta\omega_{2\gamma}+\omega_{1\gamma}\lambda_{1\beta}\tilde{C}_{1\beta}^{\gamma\alpha}((1-K^{(2)})d_2)_\alpha$$

$$-(d_1(1-K^{(1)})M(1-K^{(2)})d_2)+\omega_{1\beta}\lambda_1^\alpha S_{\alpha\gamma}^{\beta\delta}\lambda_{2\gamma}\omega_{2\delta}+(\omega_1M\Gamma_a\lambda_2)(\lambda_1\Gamma^aM\omega_2), \tag{5.45}$$

where

$$S_{\alpha\gamma}^{\beta\delta}=\frac{1}{2}C_{\alpha\gamma}^{\beta\delta}-(Y_{\alpha\eta}^{(1)\beta}\tilde{C}_{2\gamma}^{\eta\delta}-C_{1\alpha}^{\beta\eta}Y_{\eta\gamma}^{(2)\delta})+Y_{\alpha\eta}^{(1)\beta}M^{\eta\kappa}Y_{\kappa\gamma}^{(2)\delta}, \tag{5.46}$$

and

$$\lambda_1^\alpha\lambda_2^\gamma C_{\alpha\gamma}^{\beta\delta}=s_2s_1P^{\beta\delta}=-s_1s_2P^{\beta\delta}. \tag{5.47}$$

It follows from (4.42) that the fields $C_{\alpha\gamma}^{\beta\delta}$, and therefore $S_{\alpha\gamma}^{\beta\delta}$, are Lorentz-Weyl valued in α, β and in γ, δ.

Adding equations (4.39), (5.36), (5.37), (5.38) and (5.42), one obtains the pure spinor action:

$$I=I_{GS}+I_K+I_{gf}=\int[\frac{1}{2}\mathcal{E}_+^a\mathcal{E}_{-a}+(n_iB_2^i)+(\omega_1\tilde{\Delta}_+\lambda_1)+(\omega_2\tilde{\Delta}_-\lambda_2)+(\mathcal{E}_{1+}d_1)+(\mathcal{E}_{2-}d_2)$$

$$-(d_1Md_2)+d_{1\alpha}\tilde{C}_{2\beta}^{\alpha\gamma}\lambda_2^\beta\omega_{2\gamma}+\omega_{1\gamma}\lambda_{1\beta}\tilde{C}_{1\beta}^{\gamma\alpha}d_{2\alpha}+\omega_{1\beta}\lambda_1^\alpha S_{\alpha\gamma}^{\beta\delta}\lambda_{2\gamma}\omega_{2\delta}], \tag{5.48}$$

which is in full agreement with the pure spinor action first obtained by Berkovits and Howe in [11].

Notice that the field equations obtained from this action varying $d_{\underline{\alpha}}$ are

$$\mathcal{E}_1^\beta+(Md_2)^\beta-\tilde{C}_{2\alpha}^{\beta\gamma}\lambda_2^\alpha\omega_{2\gamma}=0,$$

$$\mathcal{E}_2^\beta-(d_1M)^\beta+\tilde{C}_{1\alpha}^{\beta\gamma}\lambda_1^\alpha\omega_{1\gamma}=0,$$

which justify equations (5.23) and (5.24) and assure the on shell nilpotence of s acting on $d_{\underline{\alpha}}$.

Acknowledgements

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